
Modern approaches to quantum gravity

Solution 13 Fall 2025

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1. Hawking-Page in Three Dimensions

(a) The Euclidean BTZ black hole with the identification $t_E \sim t_E + \beta$ has the metric, stated in the exercise set,

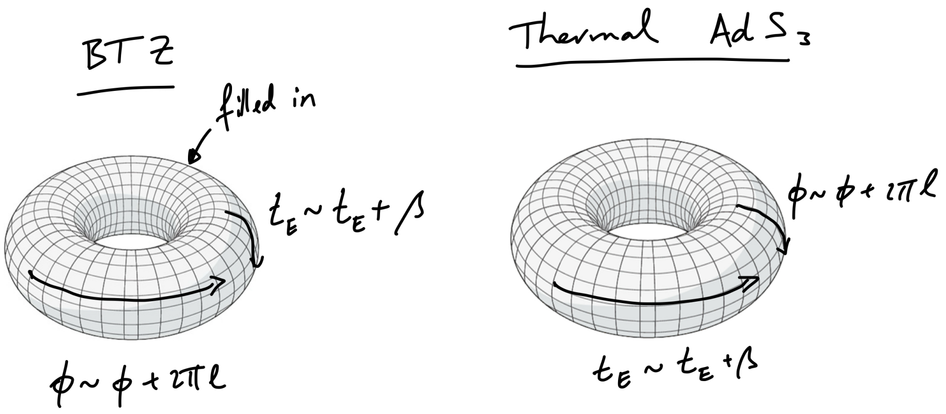
$$ds^2 = \ell^2 \left[\left(r^2 - 8M\right) dt_E^2 + \frac{dr^2}{r^2 - 8M} + r^2 d\phi^2 \right]. \tag{1}$$

Because of the identifications $\phi \sim \phi + 2\pi\ell$ and $t_E \sim t_E + \beta$, this can be understood, for a given and fixed r, as a torus. When $r \to \sqrt{8M}$, the thermal circle shrinks to 0 whereas the ϕ circle remains finite in size. Thus, we interpret this manifold as a filled in torus, where it is the thermal circle which is filled in.

The thermal AdS_3 with identification $t_E \sim t_E + \beta$ and $\phi \sim \phi + 2\pi\ell$ has the metric

$$ds^2 = \left(1 + \frac{r^2}{\ell^2}\right) dt_E^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\phi^2 \tag{2}$$

Again, we interpret it as a torus, but this time as $r \to 0$, it is the ϕ circle which shrinks while the thermal circle remains finite in size. Both manifolds are depicted in the following figure:



To compute the on-shell action of the thermal AdS manifold, we first rescale to make t_E have the same periodicity as BTZ, so that we can use its on-shell action. This means rescaling $(t_E, \phi) \to \frac{2\pi\ell}{\beta}(t_E, \phi)$, so that the filled in circle has periodicity $2\pi\ell \cdot \frac{2\pi\ell}{\beta}$, obtaining

$$S_{(th)}^{\text{on-shell}} = S_{(bh)}^{\text{on-shell}}\big|_{\beta \to \frac{4\pi^2\ell^2}{\beta}} = -\frac{\beta c}{12\ell} \tag{3}$$

(b) We are interested in the limit where $G_N \to 0$ ($\ell_p \to \infty$, $M_P \to 0$), meaning $c \to \infty$. In this limit, the partition function reduces to $Z = e^{-S_{\text{min}}^{\text{on-shell}}}$ where

$$S_{\text{min}}^{\text{on-shell}} = \min(S_{(th)}^{\text{on-shell}}, S_{(bh)}^{\text{on-shell}}) \tag{4}$$

At small temperature $T \ll 1$ ($\beta \gg 1$), the minimum is given by thermal AdS, whereas at high temperatures, it is given by BTZ. The transition takes place at

$$S_{(th)}^{\text{on-shell}} = S_{(bh)}^{\text{on-shell}} \implies T = T^* = \frac{1}{2\pi\ell} \tag{5}$$

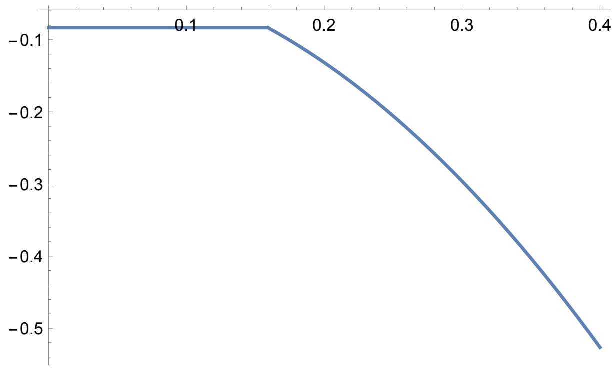
We can thus write compactly

$$S_{\text{min}}^{\text{on-shell}} = -\frac{c}{12\ell T}\Theta(T^* - T) - \frac{\pi^2 \ell c T}{3}\Theta(T - T^*) \tag{6}$$

where Θ is the Heaviside function. Thus,

$$F = -T \log Z = T S_{\text{min}}^{\text{on-shell}} = -\frac{c}{12\ell}\Theta(T^* - T) - \frac{\pi^2 \ell c T^2}{3}\Theta(T - T^*) \tag{7}$$

which can be plotted as (here I'm plotting F/c with $\ell = 1$):



Note that since $\partial F/\partial T$ is discontinuous, the transition at $T = T^*$ is a first order phase transition.

(c) The entropy is calculated from the partition function thanks to

$$S = \frac{\partial}{\partial T}(T \log Z) = -\frac{\partial}{\partial T}(T S_{\text{min}}^{\text{on-shell}}) = \frac{2\pi^2 \ell c}{3} T \, \Theta(T - T^*) \tag{8}$$

(note that the entropy vanishes for thermal AdS, as expected).

(d) To compute the mean energy in the canonical ensemble, use

$$E(\beta) = -\frac{\partial}{\partial \beta}\log Z = \frac{\partial}{\partial \beta} S_{\text{min}} = -\frac{c}{12\ell}\Theta(T^* - T) + \frac{\pi^2 \ell c T^2}{3}\Theta(T - T^*) \tag{9}$$

The first term corresponds to the Casimir energy in the CFT (in this sense, thermal AdS corresponds to the CFT vacuum), whereas the second term gives the spacetime energy $M(T)$, using the relation between M and T derived in Problem set 11.

(e) Inverting this relation, we obtain (for large T and large E where $T > T^*$)

$$1/T = \beta = \sqrt{\frac{\pi^2 c\ell}{3E}} \tag{10}$$

and thus,

$$S(E) = \frac{2\pi^2 \ell c}{3}\beta = 2\pi\sqrt{\frac{cE\ell}{3}} \tag{11}$$

(f) This gives precisely the Cardy formula, which predicts the exponentially growing density of states $\rho(E) = e^{S(E)}$ that we obtained in a previous exercise set.

2. Small and large black holes in general d

(a) We may determine the temperature of the black hole by relating it to the horizon radius r_h, which is given by the larger root of $f(r_h) = 0$, i.e.,

$$1 - \frac{\mu}{r_h^{d-2}} + \frac{r_h^2}{L^2} = 0. \tag{12}$$

Since the metric near the horizon at $r \approx r_h$ reads

$$ds^2 = f'(r_h)(r - r_h)d\tau^2 + \frac{1}{f'(r_h)(r - r_h)}dr^2 + r_h^2 d\Omega_{d-1}^2, \tag{13}$$

the temperature of the black hole is given by (recall problem set 4)

$$T = \frac{|f'(r_h)|}{4\pi} = \frac{dr_h^2 + (d-2)L^2}{4\pi L^2 r_h}. \tag{14}$$

On the other hand, the associated temperature of thermal AdS is given by the inverse of the compactified Euclidean time direction. This difference originates from the different topology of the solutions: for black holes, the thermal circle is contractible, while for thermal AdS it is not.

(b) Solving for $r_h(T)$, we find two solutions

$$r_h^{\pm} = \frac{2\pi L^2 T}{d}\left(1 \pm \sqrt{1 - \frac{d(d-2)}{4\pi^2 L^2 T^2}}\right) \tag{15}$$

and we get $T = T_{min}$ when the discriminant is 0. We see that there is no black hole solution when $T < T_{min}$. Nonetheless, thermal AdS remains a solution (it satisfies Einstein's equations with a negative cosmological constant for every T).

(c) The two solutions have been previously found

$$r_h^{\pm} = \frac{2\pi L^2 T}{d}\left(1 \pm \sqrt{1 - \frac{d(d-2)}{4\pi^2 L^2 T^2}}\right) \tag{16}$$

(d) We work in Euclidean signature with the action

$$S = S_{\text{bulk}} + S_{\text{GHY}} = -\frac{1}{2\kappa^2}\int_M d^D x\,\sqrt{g}\,(R - 2\Lambda) - \frac{1}{\kappa^2}\int_{\partial M} d^d x\,\sqrt{\gamma}\,K, \tag{17}$$

where $D = d + 1$ and $\Lambda = -\frac{(D-1)(D-2)}{2L^2}$.

For an Einstein–AdS solution we have

$$R_{\mu\nu} = -\frac{D-1}{L^2}\,g_{\mu\nu} \quad \Rightarrow \quad R = -\frac{D(D-1)}{L^2}, \tag{18}$$

so

$$R - 2\Lambda = -\frac{D(D-1)}{L^2} + \frac{(D-1)(D-2)}{L^2} = -\frac{2(D-1)}{L^2}. \tag{19}$$

Thus

$$S_{\text{bulk}} = -\frac{1}{2\kappa^2}\int \sqrt{g}\,(R-2\Lambda) = \frac{D-1}{\kappa^2 L^2}\,\mathrm{Vol}(M) = \frac{d}{\kappa^2 L^2}\,\mathrm{Vol}(M), \tag{20}$$

where $\mathrm{Vol}(M)$ is the Euclidean spacetime volume.

We now compare Euclidean AdS–Schwarzschild (BH) and thermal AdS at the same boundary temperature. The metric is

$$ds^2 = f(r)\,d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \tag{21}$$

with

$$f_{\text{BH}}(r) = 1 + \frac{r^2}{L^2} - \frac{\mu}{r^{d-2}}, \qquad f_{\text{AdS}}(r) = 1 + \frac{r^2}{L^2}. \tag{22}$$

The horizon radius r_h is defined by $f_{\text{BH}}(r_h) = 0$, which gives

$$\mu = r_h^{d-2}\left(1 + \frac{r_h^2}{L^2}\right). \tag{23}$$

We place a radial cutoff at $r = R$ and later send $R \to \infty$. The induced boundary metric at $r = R$ is

$$ds_\partial^2 = f(R)\,d\tau^2 + R^2 d\Omega_{d-1}^2. \tag{24}$$

To compare the two saddles in the same thermal ensemble, we must fix the proper length of the Euclidean time circle at $r = R$, i.e.

$$\beta_{\text{BH}}\sqrt{f_{\text{BH}}(R)} = \beta_{\text{AdS}}\sqrt{f_{\text{AdS}}(R)}. \tag{25}$$

We denote $\beta \equiv \beta_{\text{BH}}$ and thus

$$\beta_{\text{AdS}} = \beta\sqrt{\frac{f_{\text{BH}}(R)}{f_{\text{AdS}}(R)}}. \tag{26}$$

The Euclidean volumes (with cutoff R) are

$$\mathrm{Vol}_{\text{BH}}(R) = \beta\,\mathrm{Vol}(S^{d-1})\int_{r_h}^{R} dr\,r^{d-1} = \beta\,\mathrm{Vol}(S^{d-1})\frac{R^d - r_h^d}{d}, \tag{27}$$

$$\mathrm{Vol}_{\text{AdS}}(R) = \beta_{\text{AdS}}\,\mathrm{Vol}(S^{d-1})\int_{0}^{R} dr\,r^{d-1} = \beta_{\text{AdS}}\,\mathrm{Vol}(S^{d-1})\frac{R^d}{d}. \tag{28}$$

Hence

$$\Delta\mathrm{Vol}(R) = \mathrm{Vol}_{\text{BH}}(R) - \mathrm{Vol}_{\text{AdS}}(R) = \frac{\beta\,\mathrm{Vol}(S^{d-1})}{d}\left[(R^d - r_h^d) - R^d\sqrt{\frac{f_{\text{BH}}(R)}{f_{\text{AdS}}(R)}}\right]. \tag{29}$$

For large R we expand

$$\frac{f_{\text{BH}}(R)}{f_{\text{AdS}}(R)} = 1 - \frac{\mu}{R^{d-2} f_{\text{AdS}}(R)} = 1 - \varepsilon, \qquad \varepsilon \equiv \frac{\mu}{R^{d-2}(1 + R^2/L^2)}. \tag{30}$$

Then

$$\sqrt{\frac{f_{\text{BH}}(R)}{f_{\text{AdS}}(R)}} = \sqrt{1-\varepsilon} = 1 - \frac{\varepsilon}{2} + O(\varepsilon^2). \tag{31}$$

Since for large R we have $\varepsilon \sim \mu L^2/R^d$,

$$R^d \sqrt{\frac{f_{\rm BH}(R)}{f_{\rm AdS}(R)}} = R^d - \frac{\mu L^2}{2} + O(R^{d-2}). \tag{32}$$

Therefore, in the limit $R \to \infty$,

$$\Delta\mathrm{Vol} \equiv \lim_{R\to\infty} \Delta\mathrm{Vol}(R) = \frac{\beta\,\mathrm{Vol}(S^{d-1})}{d} \left(\frac{\mu L^2}{2} - r_h^d\right). \tag{33}$$

Using $S_{\rm bulk} = \frac{d}{\kappa^2 L^2}\,\mathrm{Vol}$, we obtain

$$\Delta S_{\rm bulk} = S_{\rm bulk}^{\rm BH} - S_{\rm bulk}^{\rm AdS} = \frac{d}{\kappa^2 L^2}\,\Delta\mathrm{Vol} = \frac{\beta\,\mathrm{Vol}(S^{d-1})}{\kappa^2 L^2} \left(\frac{\mu L^2}{2} - r_h^d\right). \tag{34}$$

Using $\mu = r_h^{d-2}\left(1 + \frac{r_h^2}{L^2}\right)$, we obtain

$$\frac{\mu L^2}{2} - r_h^d = \frac{r_h^{d-2}}{2}\left(L^2 + r_h^2\right) - r_h^d = \frac{r_h^{d-2}}{2}\left(L^2 + r_h^2 - 2r_h^2\right) = \frac{r_h^{d-2}}{2}\left(L^2 - r_h^2\right). \tag{35}$$

Thus the Euclidean bulk action difference is

$$\Delta S_{\rm bulk} = \frac{\beta\,\mathrm{Vol}(S^{d-1})}{2\kappa^2 L^2}\, r_h^{d-2}\left(L^2 - r_h^2\right). \tag{36}$$

We now compute the Gibbons–Hawking–York term. For the metric

$$ds^2 = f(r)\,d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \tag{37}$$

the induced metric at $r = \text{const}$ is

$$\gamma_{ij}dx^i dx^j = f(r)\,d\tau^2 + r^2 d\Omega_{d-1}^2, \qquad \sqrt{\gamma} = r^{d-1}\sqrt{f(r)}\,\sqrt{\det g_{S^{d-1}}}. \tag{38}$$

The outward unit normal is $n^\mu = (0, \sqrt{f}, 0, \dots)$, and one finds

$$K(r) = \sqrt{f(r)}\left(\frac{f'(r)}{2f(r)} + \frac{d-1}{r}\right), \tag{39}$$

so that

$$\sqrt{\gamma}\,K = r^{d-1}\left(\frac{f'(r)}{2} + (d-1)\frac{f(r)}{r}\right)\sqrt{\det g_{S^{d-1}}}. \tag{40}$$

Define

$$C[f](R) \equiv r^{d-1}\left(\frac{f'(r)}{2} + (d-1)\frac{f(r)}{r}\right)\Big|_{r=R}. \tag{41}$$

Then

$$S_{\rm GHY}(R) = -\frac{1}{\kappa^2}\int_{\partial M} d^dx\,\sqrt{\gamma}\,K = -\frac{1}{\kappa^2}\,\beta\,\mathrm{Vol}(S^{d-1})\,C[f](R), \tag{42}$$

with β the appropriate Euclidean period for the given geometry.

For thermal AdS ($\mu = 0$),

$$f_{\rm AdS}(r) = 1 + \frac{r^2}{L^2}, \qquad f'_{\rm AdS}(r) = \frac{2r}{L^2}, \tag{43}$$

so

$$C_{\rm AdS}(R) = (d-1)R^{d-2} + d\,\frac{R^d}{L^2}. \tag{44}$$

For the black hole,

$$f_{\rm BH}(r) = 1 + \frac{r^2}{L^2} - \frac{\mu}{r^{d-2}}, \qquad f'_{\rm BH}(r) = \frac{2r}{L^2} + (d-2)\frac{\mu}{r^{d-1}}, \tag{45}$$

and one finds

$$C_{\rm BH}(R) = (d-1)R^{d-2} + d\,\frac{R^d}{L^2} - \frac{d\,\mu}{2}. \tag{46}$$

In particular,

$$C_{\rm BH}(R) - C_{\rm AdS}(R) = -\frac{d\,\mu}{2}, \tag{47}$$

which is independent of R.

The GHY difference at cutoff R is therefore

$$\begin{aligned}
\Delta S_{\rm GHY}(R) &= S^{\rm BH}_{\rm GHY}(R) - S^{\rm AdS}_{\rm GHY}(R) &(48)\\
&= -\frac{\mathrm{Vol}(S^{d-1})}{\kappa^2}\left[\beta\, C_{\rm BH}(R) - \beta_{\rm AdS}\, C_{\rm AdS}(R)\right] &(49)\\
&= -\frac{\mathrm{Vol}(S^{d-1})}{\kappa^2}\Big[\beta\big(C_{\rm AdS}(R) - \tfrac{d\mu}{2}\big) - \beta_{\rm AdS}\, C_{\rm AdS}(R)\Big] &(50)\\
&= -\frac{\mathrm{Vol}(S^{d-1})}{\kappa^2}\left[(\beta-\beta_{\rm AdS})C_{\rm AdS}(R) - \beta\,\frac{d\mu}{2}\right]. &(51)
\end{aligned}$$

We now use the matching condition

$$\beta_{\rm AdS} = \beta\sqrt{\frac{f_{\rm BH}(R)}{f_{\rm AdS}(R)}} = \beta\left(1 - \frac{\varepsilon}{2} + O(\varepsilon^2)\right), \qquad \varepsilon = \frac{\mu}{R^{d-2}(1+R^2/L^2)}, \tag{52}$$

so that

$$\beta - \beta_{\rm AdS} = \beta\,\frac{\varepsilon}{2} + O(\varepsilon^2). \tag{53}$$

Using $C_{\rm AdS}(R) = (d-1)R^{d-2} + dR^d/L^2$ and the definition of ε, one finds

$$(\beta-\beta_{\rm AdS})C_{\rm AdS}(R) - \beta\,\frac{d\mu}{2} = -\,\beta\,\frac{\mu L^2}{2(L^2+R^2)} + O\!\left(\frac{1}{R^2}\right). \tag{54}$$

Thus

$$\Delta S_{\rm GHY}(R) = \frac{\beta\,\mathrm{Vol}(S^{d-1})}{\kappa^2}\,\frac{\mu L^2}{2(L^2+R^2)} + O\!\left(\frac{1}{R^2}\right) \xrightarrow{R\to\infty} 0. \tag{55}$$

We conclude that the finite contribution from the GHY term vanishes

$$\Delta S_{\rm GHY} = 0. \tag{56}$$

In the Euclidean path integral, the free energy is

$$F = T\, S_{\text{on-shell}} = \frac{1}{\beta}\, S_{\text{on-shell}}, \tag{57}$$

and therefore

$$\Delta F = \frac{\mathrm{Vol}(S^{d-1})}{2\kappa^2 L^2} r_h^{d-2} \left(L^2 - r_h^2\right) = \frac{\mathrm{Vol}(S^{d-1})}{2\kappa^2} r_h^{d-2} \left(1 - \frac{r_h^2}{L^2}\right). \tag{58}$$

This is precisely the desired expression for the free energy difference between the AdS–Schwarzschild black hole and thermal AdS at fixed boundary temperature $T = 1/\beta$.

(e) For $r_h < L$, the difference ΔF of the free energies is positive, and therefore thermal AdS is preferred. For $r_h > L$, the Schwarzschild black hole is preferred. For $r_h = L$, there is a phase transition, the Hawking–Page phase transition, with transition temperature

$$T_{HP} = \frac{1}{2\pi L}(d-1).$$

Small black holes have $r_h < L$, hence their free energy is always greater than thermal AdS, and such configurations never dominate the partition function.

(f) The specific heat can be related to the free energy by

$$C = T\frac{d^2F}{dT^2} \tag{59}$$

To conclude, we see that large black holes are stable, while small ones decay into thermal AdS for $T \leq \frac{d-1}{2\pi L}$ and into large black holes for $T > \frac{d-1}{2\pi L}$.

(g) As we take $L \to \infty$, we recover the flat space limit, where $T_{HP} = 0$, hence no phase transition occurs, and the partition function is always dominated by the black hole solution, of which there is now a single one, with free energy $F = \frac{r_h^{d-2}}{2\kappa^2}\mathrm{Vol}(S^{d-1})$. This is always positive and therefore black holes will tend to decay to the empty flat space solution, with $F = 0$.